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Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications

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Abstract

Let A and B be positive invertible operators. Then for each $p \geq 0$ and $r \geq 0$, two inequalities

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ and } A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$$

are equivalent. In this report, we shall show relations between these inequalities in case A and B are not invertible. And we shall show some applications of this result to operator classes.

1 Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be *positive* (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$.

As a recent development on order preserving operator inequalities, it is known the following Theorem F.

Theorem F (Furuta inequality [9]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\qquad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\qquad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

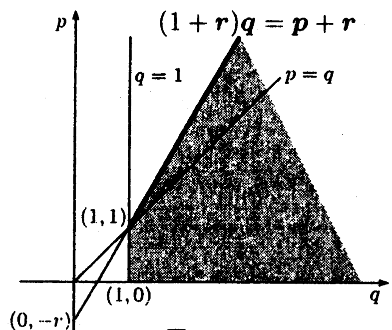

FIGURE 1

Theorem F yields the famous Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$*" by putting $r = 0$ in (i) or (ii) of Theorem F. Alternative proofs of Theorem F are given in [6] and [18] and also an elementary one page proof in [10]. It was shown by Tanahashi [19] that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem F.

As an application of Theorem F, the following result was shown in [7] and [11].

Theorem FC ([7][11]). *Let $A, B > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

(iii) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p \geq 0$ *and* $r \geq 0$.

We remark that this result is an extension of [4] in case $p = r$, and an excellent proof of this result which used only Theorem F was shown in [22].

On the other hand, the following assertions are well known: Let A and B be positive invertible operators. Then

(1) $A \geq B \Longrightarrow \log A \geq \log B$.

(2) $\log A \geq \log B \Longrightarrow (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ for all $p \geq 0$ and $r \geq 0$.

(3) For each $p \geq 0$ and $r \geq 0$, $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Longleftrightarrow A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(1) holds since $\log t$ is an operator monotone function. (2) is an immediate consequence of Theorem FC. (3) was shown in [11].

Related to these results, it is known in [23] that invertiblity of (1) and (2) can be replaced with the condition $N(A) = N(B) = \{0\}$, that is, (1) and (2) hold for some non-invertible operators A and B. But we have not known whether invertiblity of A and B in (3) can be replaced with looser condition or not. In this report, we shall show relations between

$$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \quad \text{and} \quad A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$$

when A and B are not invertible.

Next, An operator T is said to be ***hyponormal*** if $T^*T \geq TT^*$. An operator T is *invertible* log-*hyponormal* (defined in [20]) if $\log T^*T \geq \log TT^*$. For each $s > 0$ and $t > 0$, an operator T belongs to *class* $A(s,t)$ (defined in [8]) if $(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$, where $|T| = (T^*T)^{\frac{1}{2}}$. Class A$(s,t)$ is introduced as a generalization of *class A*

($|T^2| \geq |T|^2$) defined in [14]. We remark that class A equals class A$(1,1)$ and class A is introduced as a class of operators including invertible log-hyponormal operators and included in the class of *paranormal* operators ($\|T^2x\| \geq \|Tx\|^2$ for all unit vectors $x \in H$). Moreover, for each $s > 0$ and $t > 0$, an operator T belongs to *class* $wA(s,t)$ (defined in [16]) if $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$. Obviously, for each $s > 0$ and $t > 0$, every class wA(s,t) operator belongs to class A(s,t). As inclusion relations among these classes, the following assertions hold by (1), (2) and (3):

(1)' *Every invertible hyponormal operator is* log-*hyponormal.*

(2)' *Every invertible* log-*hyponormal operator belongs to class* $wA(s,t)$ *for all* $s > 0$ *and* $t > 0$.

(3)' *For each* $s > 0$ *and* $t > 0$, *invertible class* $wA(s,t)$ *equals invertible class* $A(s,t)$.

There are many papers on these classes in case of invertible operators, for example [8], [20] and [24].

On the other hand, even if an operator is non-invertible, log-hyponormality can be defined by $N(T^*) \supset N(T)$ and $\log A \geq \log B$, where A and B are the compressions of T^*T and TT^* to $\overline{R(T)}$, respectively. This definition implicitly appeared in [3] and it was pointed out in [23] that it is the general form of log-hyponormality. Ando [3] showed that every hyponormal operator is log-hyponormal and every log-hyponormal operator is paranormal. Moreover, Uchiyama [23] showed that every log-hyponormal operator is also included in class A (even if an operator is non-invertible). In this report, we shall show that for each $s > 0$ and $t > 0$, class A(s,t) coincides with class wA(s,t), that is, we shall show (3)' without invertibility of operators, and show some properties of class A(s,t) operators. Lastly, we shall show a normality of class A(s,t) operators for $s > 0$ and $t > 0$.

2 Relations between $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$

In this section, we shall show the following result:

Theorem 1. *Let* A *and* B *be positive operators. Then for each* $p \geq 0$ *and* $r \geq 0$, *the following assertions hold:*

(i) *If* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, *then* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.

(ii) *If* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and* $N(A) \subset N(B)$, *then* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

We remark on Theorem 1 that the assumption of (ii) has a kernel condition $N(A) \subset N(B)$, but the assumption of (i) does not have any kernel conditions. If A and B are invertible, then $N(A) = N(B) = \{0\}$ holds, and the kernel condition of (ii) in Theorem 1 is satisfied. Hence we know that Theorem 1 is a generalization of (3) in the previous section.

To prove Theorem 1, we prepare the following lemma.

Lemma 2. *Let A and B be positive operators. Then the following assertions hold:*

(i) $\lim_{\varepsilon\to+0} A^{\frac{1}{2}}(A+\varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon\to+0}(A+\varepsilon I)^{-1}A = P_{N(A)^{\perp}}$,
where $P_{\mathcal{M}}$ is the projection onto a closed subspace $\mathcal{M}$.

(ii) $\lim_{\varepsilon\to+0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha}+\varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$ *for* $\alpha \in (0,1)$.

We remark that if A and B are both positive invertible, then

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha} \text{ for } \alpha \in (0,1)$$

by the following Lemma F. Therefore we can regard (ii) of Lemma 2 as a non-invertible version of Lemma F for $\lambda \in (0,1)$.

Lemma F **([12]).** *Let A be a positive invertible operator and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Proof of Lemma 2.
Proof of (i). We give a proof which is a slightly modification of the proof of [5, Lemma]. Let $A = \int_0^{\|A\|} t\,dF(t)$ be the spectral decomposition of A. Then

$$\lim_{\varepsilon\to+0}(A+\varepsilon I)^{-1}A = \lim_{\varepsilon\to+0}\int_0^{\|A\|}\frac{t}{t+\varepsilon}\,dF(t) = \int_0^{\|A\|}\chi_{(0,\|A\|]}(t)\,dF(t) = I - F(0),$$

where $\chi_{(0,\|A\|]}(t)$ is the characteristic function on $(0,\|A\|]$. Since $I - F(0) = P_{N(A)^{\perp}}$, we have

$$\lim_{\varepsilon\to+0} A^{\frac{1}{2}}(A+\varepsilon I)^{-1}A^{\frac{1}{2}} = \lim_{\varepsilon\to+0}(A+\varepsilon I)^{-1}A = P_{N(A)^{\perp}}.$$

Proof of (ii). Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition of $A^{\frac{1}{2}}B^{\frac{1}{2}}$.

Then we have

$$\begin{aligned}
&\lim_{\varepsilon\to+0} A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha}+\varepsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}\\
&=\lim_{\varepsilon\to+0} U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\alpha}(|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2\alpha}+\varepsilon I)^{-1}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{\alpha}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^*\\
&=U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}P_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|)^{\perp}}|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{1-\alpha}U^* \qquad \text{by (i)}\\
&=U|A^{\frac{1}{2}}B^{\frac{1}{2}}|^{2(1-\alpha)}U^*=|B^{\frac{1}{2}}A^{\frac{1}{2}}|^{2(1-\alpha)}=(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}.
\end{aligned}$$

Hence the proof is complete. □

Proof of Theorem 1. Let $\varepsilon>0$. And also we may assume $p>0$ and $r>0$.
Proof of (i). Since $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}\ge B^r$, we obtain

$$A^{\frac{p}{2}}B^{\frac{r}{2}}(B^r+\varepsilon I)^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}\ge A^{\frac{p}{2}}B^{\frac{r}{2}}\{(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}+\varepsilon I\}^{-1}B^{\frac{r}{2}}A^{\frac{p}{2}}. \tag{2.1}$$

In (2.1), by tending $\varepsilon\to+0$ and Lemma 2, we obtain

$$A^{\frac{p}{2}}P_{N(B)^{\perp}}A^{\frac{p}{2}}\ge(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Hence we have

$$A^p\ge A^{\frac{p}{2}}P_{N(B)^{\perp}}A^{\frac{p}{2}}\ge(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Proof of (ii). Since $A^p\ge(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$, we obtain

$$B^{\frac{r}{2}}A^{\frac{p}{2}}(A^p+\varepsilon I)^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}}\le B^{\frac{r}{2}}A^{\frac{p}{2}}\{(A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}+\varepsilon I\}^{-1}A^{\frac{p}{2}}B^{\frac{r}{2}}. \tag{2.2}$$

In (2.2), by tending $\varepsilon\to+0$ and Lemma 2, we obtain

$$B^{\frac{r}{2}}P_{N(A)^{\perp}}B^{\frac{r}{2}}\le(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}. \tag{2.3}$$

On the other hand,

$$N(A)\subset N(B)\iff P_{N(A)^{\perp}}\ge P_{N(B)^{\perp}}. \tag{2.4}$$

By (2.3) and (2.4), we have

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}\ge B^{\frac{r}{2}}P_{N(A)^{\perp}}B^{\frac{r}{2}}\ge B^{\frac{r}{2}}P_{N(B)^{\perp}}B^{\frac{r}{2}}=B^r.$$

Therefore the proof is complete. □

Remark. We recall the assumptions of (i) and (ii) in Theorem 1. Here we assume $p = r = 1$ in Theorem 1.

(i-a) $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$.

(ii-a) $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ *and* $N(A) \subset N(B)$.

We proved that (i-a) ensures $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and (ii-a) ensures (i-a) in Theorem 1, so we might expect that (i-a) and (ii-a) are equivalent. But we have the following counterexample.

Example 1. $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$ and $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, but $N(A) \not\subset N(B)$.

Let $A = 2\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^{\frac{1}{2}} = \sqrt{\frac{2}{5}}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $B^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = B$. Hence

$$\sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$$

and

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \frac{\sqrt{2}}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

But $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \in N(A)$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix} \notin N(B)$, so that $N(A) \not\subset N(B)$.

Moreover, we have the following example in [16].

Example 2 ([16]). $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, but $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$ and $N(A) \not\subset N(B)$.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can check $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$, $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$ and $N(A) \not\subset N(B)$, easily.

Therefore we recognize that $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ requires some condition to be equivalent to $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$. So we consider the following condition.

(ii-a') $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ *and* $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$.

We can easily check that $N(A) \subset N(B)$ ensures $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$. And also $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \geq B$ ensures $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$ since $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) = N(B^{\frac{1}{2}}AB^{\frac{1}{2}}) = N((B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}}) \subset N(B)$, so that (i-a) ensures (ii-a') by (i) in Theorem 1.

But, unfortunately, we understand that (ii-a') does not ensure (i-a) by the following example.

Example 3. $A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}$ and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) \subset N(B)$, but $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B$.

Let $A = \frac{1}{2}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$, $B = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Then $A^{\frac{1}{2}} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$, $B^{\frac{1}{2}} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = B$. Hence

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}} = \frac{1}{\sqrt{50}}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$$

and $N(A^{\frac{1}{2}}B^{\frac{1}{2}}) = N(B) = \left\{t\begin{pmatrix}0\\ 1\end{pmatrix} : t \in \mathbb{C}\right\}$ since $A^{\frac{1}{2}}B^{\frac{1}{2}} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}$. But

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = (B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{2}} \not\geq B.$$

At the end of this remark, we note that Chō-Huruya-Kim [5] gave an example such that $N(T) \not\subset N(T^*)$, $N(T) \not\supset N(T^*)$ and $|\widetilde{T}| \geq |T| \geq |(\widetilde{T})^*|$ (i.e., T is w-hyponormal) by using A and B in Example 1 stated above, where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ and $T = U|T|$ is the polar decomposition of T.

3 Applications

In this section, we shall show some applications of Theorem 1 to operator classes. In section 1, we introduced definitions of some operator classes, here we recall definitions of these classes as follows:

Definition 1. *Let $s > 0$, $t > 0$ and $T = U|T|$ be the polar decomposition of T.*

(i) *T belongs to class $A(s,t)$ $\Longleftrightarrow$ $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$.*

(ii) *T belongs to class $wA(s,t)$*
$\Longleftrightarrow$ $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$
$\Longleftrightarrow$ $|\widetilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t}$ and $|T|^{2s} \geq |(\widetilde{T}_{s,t})^*|^{\frac{2s}{s+t}}$,
where $\widetilde{T}_{s,t} = |T|^sU|T|^t$ (generalized Aluthge transformation).

(iii) *T belongs to class A $\Longleftrightarrow$ $|T^2| \geq |T|^2$, that is, T belongs to class $A(1,1)$.*

(iv) *T is w-hyponormal $\Longleftrightarrow$ $|\widetilde{T}| \geq |T| \geq |(\widetilde{T})^*|$, that is, T belongs to class $wA(\frac{1}{2},\frac{1}{2})$,*
where $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (Aluthge transformation).

(i), (ii), (iii) and (iv) of Definition 1 were defined in [8], [16], [14] and [2], respectively. We remark that Aluthge transformation has many interesting properties, and many authors study this transformation, for instance, [1], [13], [15] and [17]. These classes include invertible log-hyponormal operators, and are included in normaloid (i.e., $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T). It has been known that for each $s > 0$ and $t > 0$, class $\mathrm{A}(s,t)$ includes class $w\mathrm{A}(s,t)$ by the definitions (i) and (ii). And also for each $s > 0$ and $t > 0$, every invertible class $\mathrm{A}(s,t)$ operator is an invertible class $w\mathrm{A}(s,t)$ operator, which was shown in [8] and [16]. More precise inclusion relations among class $w\mathrm{A}(s,t)$, and powers of class $w\mathrm{A}(s,t)$ operators were already shown as follows:

Theorem A ([16], [26]).

(i) *For each $s > 0$ and $t > 0$, every class $wA(s,t)$ operator is a class $wA(\alpha,\beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.*

(ii) *Let T be a class $wA(s,t)$ operator for $s \in (0,1]$ and $t \in (0,1]$. Then for each natural number n, T^n belongs to class $wA(\frac{s}{n}, \frac{t}{n})$.*

We remark that Theorem A holds for classes of invertible class $\mathrm{A}(s,t)$ operators instead of class $w\mathrm{A}(s,t)$, which were shown in [8] and [25]. We can summarize inclusion relations among these classes as the following Figure 2. Dotted lines in the diagram mean that we need invertibility of operators to prove the relations.

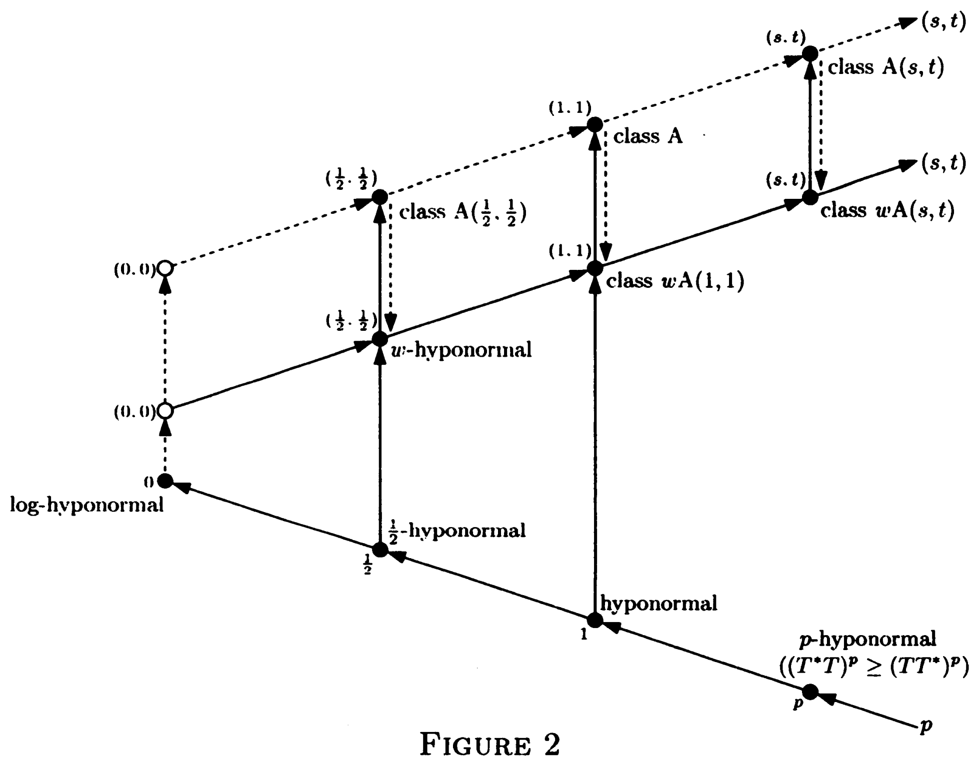

FIGURE 2

Here, in general, we can obtain that class $\mathrm{A}(s,t)$ coincides with class $w\mathrm{A}(s,t)$ by (i) of Theorem 1 as follows:

Theorem 3. *For each $s > 0$ and $t > 0$, the following assertions hold:*

(i) *Class $A(s,t)$ coincides with class $wA(s,t)$.*

(ii) *Class A coincides with class $wA(1,1)$.*

(iii) *Class $A(\frac{1}{2},\frac{1}{2})$ coincides with the class of w-hyponormal operators, i.e., class $wA(\frac{1}{2},\frac{1}{2})$.*

We can prove Theorem 3 by only applying (i) of Theorem 1 to definitions of these classes, so we omit to prove. By (iii) of Theorem 3, we have

$$|\widetilde{T}| \geq |T| \iff (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \iff T : \text{class A}(\tfrac{1}{2},\tfrac{1}{2})$$
$$\iff T : w\text{-hyponormal} \iff |\widetilde{T}| \geq |T| \geq |(\widetilde{T})^*|.$$

Hence

$$|\widetilde{T}| \geq |T| \implies |T| \geq |(\widetilde{T})^*|,$$

that is, we may as well define w-hyponormality by only $|\widetilde{T}| \geq |T|$.

Next, we shall show some properties of class $\mathrm{A}(s,t)$ operators without the assumption of invertibility, which are known as properties of invertible class $\mathrm{A}(s,t)$ operators and class $w\mathrm{A}(s,t)$ operators.

Theorem 4.

(i) *For each $s > 0$ and $t > 0$, every class $A(s,t)$ operator is a class $A(\alpha,\beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.*

(ii) *Let T be a class $A(s,t)$ operator for $s \in (0,1]$ and $t \in (0,1]$. Then for each natural number n, T^n belongs to class $A(\frac{s}{n},\frac{t}{n})$.*

Proof is very easy by (i) of Theorem 3 and Theorem A, so we omit the proof, too. By putting $s = t = 1$ in (ii) of Theorem 4 and noting that class $\mathrm{A}(\frac{1}{2},\frac{1}{2})$ equals w-hyponormality by (iii) of Theorem 3, we obtain the following result on powers of class A operators without the assumption of invertibility.

Corollary 5. *Let T be a class A operator. Then for each natural number n, T^n belongs to class $A(\frac{1}{n},\frac{1}{n})$. Especially T^2 is w-hyponormal.*

At the end of this section, we shall summarize relations among these classes which are obtained in this section as follows: Please compare Figure 3 with Figure 2 stated

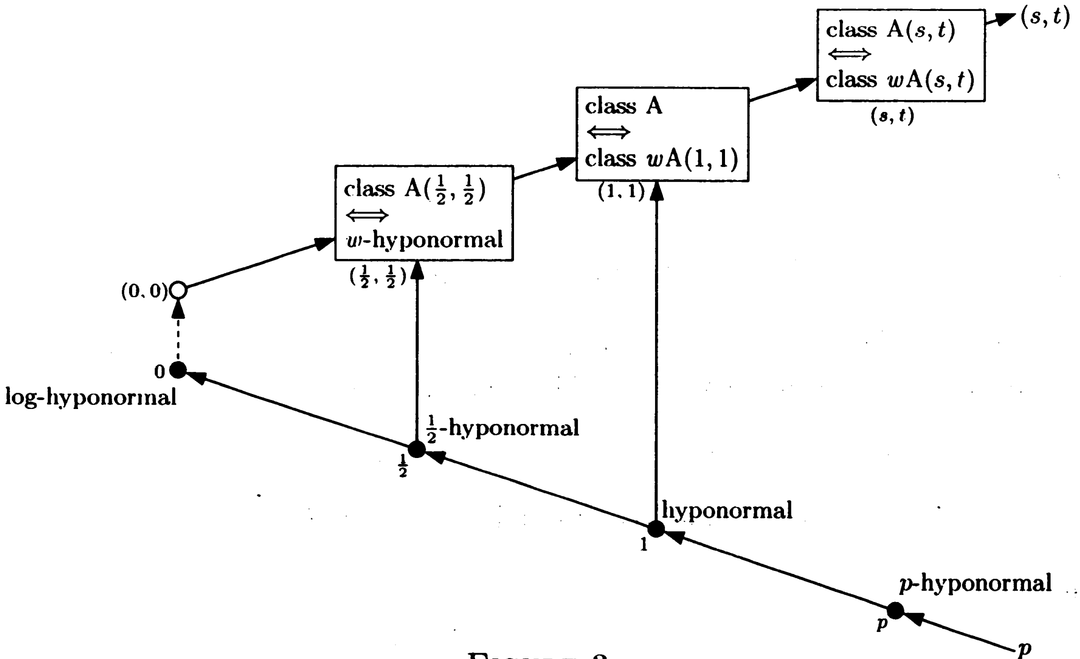

FIGURE 3

4 Normality

In this section, we shall show a normality of some non-normal operators. It is well known that if T and T^* are hyponormal, then T is normal. But in the case T and T^* belong to weaker class than hyponormal, this assertion is not obvious. Many authors obtained many results on this problem, and the following results were known until now.

Theorem B **([21]).** *If T is a class A operator and T^* is a w-hyponormal operator, then T is normal.*

Theorem C **([3]).** *If T and T^* are paranormal operators satisfying $N(T) = N(T^*)$, then T is normal.*

Here, we shall generalize Theorem B as follows:

Theorem 6. *Let $s_1 > 0$, $s_2 > 0$, $t_1 > 0$ and $t_2 > 0$. If T belongs to class $A(s_1, t_1)$ and T^* belongs to class $A(s_2, t_2)$, then T is normal.*

Put $s_1 = t_1 = 1$ and $s_2 = t_2 = \frac{1}{2}$ in Theorem 6, we have Theorem B by Theorem 3, put $s_1 = t_1 = s_2 = t_2 = 1$ in Theorem 6, we obtain the following result on class A:

Corollary 7. *If T and T^* belong to class A, then T is normal.*

To prove Theorem 6, we need the following results:

Lemma 8. *Let A and B be self-adjoint operators, and $X \in B(H)$ satisfying*

$$X^*AX \geq X^*BX.$$

If $N(A) \supset N(X^*)$ and $N(B) \supset N(X^*)$, then $A \geq B$.

Proof. Let $H = \overline{R(X)} \oplus N(X^*)$ and $x = Xy + z$, where $y \in H$ and $z \in N(X^*)$. Put $T = A - B$. Then $T = T^*$ and $N(T) \supset N(X^*)$. Hence we have

$$(Tx, x) = (TXy, Xy) + (TXy, z) + (Tz, Xy) + (Tz, z) = (X^*TXy, y) \geq 0,$$

that is, $A \geq B$. □

Proposition 9. *Let $A \geq 0$ and $B \geq 0$. If*

$$B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \tag{4.1}$$

and

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2, \tag{4.2}$$

then $A = B$.

Proof. Put $E = P_{N(A)^\perp}$ and $F = P_{N(B)^\perp}$. (4.1) is equivalent to $B^{\frac{1}{2}}FAFB^{\frac{1}{2}} \geq B^2 = B^{\frac{1}{2}}BB^{\frac{1}{2}}$. By Lemma 8, we have $FAF \geq B$ since $N(FAF) \supset N(B^{\frac{1}{2}})$ and $N(B) = N(B^{\frac{1}{2}})$. Then we have

$$(A^{\frac{1}{2}}FA^{\frac{1}{2}})^2 = A^{\frac{1}{2}}FAFA^{\frac{1}{2}} \geq A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2 \quad \text{by (4.2)},$$

and we obtain the following (4.3) by Löwner-Heinz theorem.

$$A^{\frac{1}{2}}FA^{\frac{1}{2}} \geq A. \tag{4.3}$$

(4.3) is equivalent to $A^{\frac{1}{2}}EFEA^{\frac{1}{2}} \geq A^{\frac{1}{2}}EA^{\frac{1}{2}}$. By Lemma 8, we have $EFE \geq E$ since $N(EFE) \supset N(A^{\frac{1}{2}})$ and $N(E) = N(A^{\frac{1}{2}})$. Therefore we obtain $EFE = E$, and then $F \geq E$, so that $N(A) \supset N(B)$. Hence

$$A \geq B$$

by applying Lemma 8 to (4.1).

By the same way, we also get $B \geq A$, so that $A = B$. □

Proof of Theorem 6. Let $p = \max\{s_1, s_2, t_1, t_2\}$.

Firstly, if T belongs to class $\mathrm{A}(s_1, t_1)$, then T belongs to class $\mathrm{A}(p, p)$ by (i) of Theorem 4. This class coincides with class $w\mathrm{A}(p, p)$ by (i) of Theorem 3. Hence we have

$$(|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}} \geq |T^*|^{2p} \quad \text{and} \quad |T|^{2p} \geq (|T|^p|T^*|^{2p}|T|^p)^{\frac{1}{2}}. \tag{4.4}$$

Secondly, if T^* belongs to class $\mathrm{A}(s_2, t_2)$, then T^* belongs to class $\mathrm{A}(p,p)$ by (i) of Theorem 4. This class coincides with class $w\mathrm{A}(p,p)$ by (i) of Theorem 3. Hence we have

$$(|T|^p|T^*|^{2p}|T|^p)^{\frac{1}{2}} \geq |T|^{2p} \quad \text{and} \quad |T^*|^{2p} \geq (|T^*|^p|T|^{2p}|T^*|^p)^{\frac{1}{2}}. \tag{4.5}$$

Therefore

$$|T|^p|T^*|^{2p}|T|^p = |T|^{4p} \quad \text{and} \quad |T^*|^{4p} = |T^*|^p|T|^{2p}|T^*|^p$$

hold by (4.4) and (4.5), and then $|T| = |T^*|$ by Proposition 9. □

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